A Universal Circuit for Studying and Generating Chaos—Part I: Routes to Chaos

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***Abstract*—In this introductory tutorial paper, we demonstrate the generality of Chua's oscillator in generating chaos and bifurcation phenomena by electronic laboratory experiments which illustrate the standard routes to chaos, and by giving a result which shows that Chua's oscillator can generate the same qualitative behavior as any member of a 21-parameter family $\mathcal{C}$ of continuous, odd-symmetric, piecewise-linear vector field in $\mathbb{R}^3$. This result is of fundamental importance because it unifies many previously published papers on chaotic circuits and systems (e.g. examples from Brockett, Sparrow, Arnéodo, Nishio, Ogorzalek, etc.) under one umbrella, thereby obviating the need to analyze these circuits and systems as separate and unrelated systems. Indeed, every bifurcation and chaotic phenomena exhibited by any member of the family $\mathcal{C}$ is also exhibited by this universal circuit. In a companion paper [1], we show how the generality of Chua's oscillator can be used to approximate other chaotic systems in the literature which are not necessarily piecewise-linear.**

I. Introduction

CHUA'S circuit (Fig. 1) is a nonlinear electronic circuit that is the object of much scientific research activities. This circuit contains four linear elements (two capacitors, one inductor, and one resistor) and a nonlinear resistor, called *Chua's diode* [2], which can be built using off-the-shelf op-amps.

Since Chua's circuit is endowed with an unusually rich repertoire of nonlinear dynamical phenomena, it has become a *universal paradigm* for chaos. By adding a linear resistor in series with the inductor, we obtain Chua's oscillator [3], shown in Fig. 2. This circuit can generate even more chaotic phenomena and is *canonical* in the sense that its vector field is *topologically conjugate* (i.e. qualitatively equivalent) to a large class of 3-D vector fields. In this tutorial paper, we will illustrate some of the phenomena that occur in Chua's oscillator. In particular, we show some classic chaotic phenomena that have been found, such as *period doubling*, *intermittency*, and *torus breakdown*. We will adopt Madan's terminology [4, Editorial] in describing Chua's oscillator and its attractors.

The state equations of Chua's oscillator are:

$$\frac{dv_1}{dt} = \frac{1}{C_1}[G(v_2 - v_1) - f(v_1)]$$

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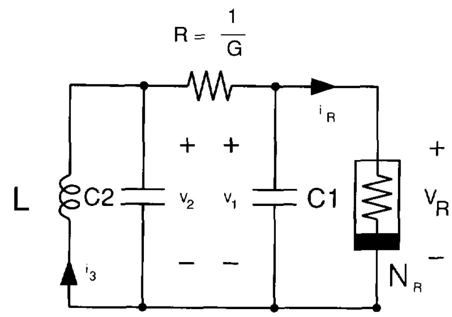

Fig. 1. Chua's circuit.

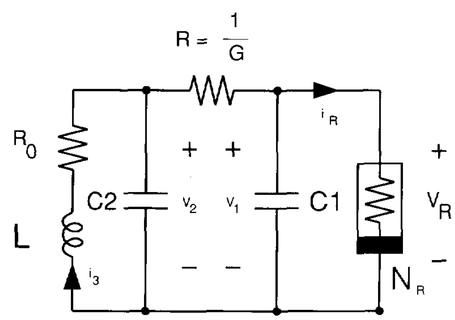

Fig. 2. Chua's oscillator.

$$\frac{dv_2}{dt} = \frac{1}{C_2}[G(v_1 - v_2) + i_3]$$
$$\frac{di_3}{dt} = -\frac{1}{L}(v_2 + R_0 i_3) \tag{1}$$

where

$$G = \frac{1}{R}$$

and

$$f(v_1) = G_b v_1 + \tfrac{1}{2}(G_a - G_b)\{|v_1 + E| - |v_1 - E|\} \tag{2}$$

is the $v - i$ characteristic of the nonlinear resistor N_R with a slope equal to G_a in the inner region and G_b in the outer region. A typical $v - i$ characteristic of N_R is shown in Fig. 3. By choosing appropriate values for G_a, G_b, and E, *any* continuous three-segment odd-symmetric piecewise-linear $v{-}i$ characteristic for Chua's diode can be specified.

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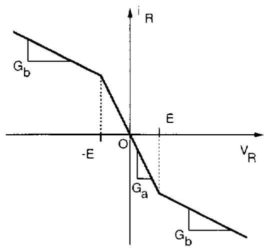


Fig. 3. Typical v-i characteristic of Chua's diode.

II. Experimental Chaos from Chua's Oscillator

For a fixed set of parameters, the Chua's oscillator equations (1) define a dynamical system which behaves in a certain manner. For example, the trajectories could converge to an equilibrium point, a limit cycle, or a strange attractor. An equilibrium point could be stable or unstable. We are interested in changes in the qualitative behavior of the system, or *bifurcations*, as parameters are varied. In this section, we demonstrate experimentally several chaotic phenomena and bifurcations occurring in Chua's oscillator:

1. Period-doubling route to chaos
2. intermittency route to chaos
3. period-adding bifurcations
4. attractors in Chua's oscillator whose only *active* element is a *linear* resistor
5. torus breakdown route to chaos.

Several visual aids that we will utilize to visualize chaos and bifurcation phenomena are the following:

1. Time-waveform of state variables
2. phase portrait of trajectories
3. Poincaré or first-return maps
4. bifurcation diagrams.

2.1. Period-Doubling Route to Chaos

In our experimental setup, the resistor value R is varied as a parameter and the following values are fixed:

$$C_1 = 5.75nF, C_2 = 21.32nF, L = 12mH, R_0 = 30.86\Omega$$

$$G_a = -0.879mS, \quad G_b = -0.4124mS, \quad E = 1V$$

While any circuit which realizes the nonlinear $v - i$ characteristic of Chua's diode can be used in the experiments reported here, we use the dual op-amp circuit of Kennedy [2].

In this route to chaos, an equilibrium point loses stability and a stable limit cycle emerges through an Andronov-Hopf bifurcation[1] when the resistance R is decreased. As the value of R is further decreased, the stable limit cycle eventually loses stability, and a stable limit cycle of approximately twice the period emerges, which we shall refer to as a period-2 limit cycle. As R is decreased further, this period-2 limit cycle in turn loses stability, and a stable period-4 limit cycle appears. This bifurcation occurs infinitely many times at ever-decreasing intervals of resistance parameter range which converges at a geometric rate to a limit (bifurcation point) at which point chaos is observed. This is illustrated in Fig. 4(a)–(e), where we start with a stable equilibrium point (Fig. 4(a)), and followed by a stable limit cycle (Fig. 4(b)), a stable period-2 limit cycle (Fig. 4(c)), a stable period-4 limit cycle (Fig. 4(d)) and finally a spiral Chua's attractor (Fig. 4(e)).

2.2. Intermittency Route to Chaos

Using the same parameters as in the previous section, when R is decreased further beyond our first chaotic phenomenon arising from period-doubling, we find a range of parameter values where a stable period-3 limit cycle is observed. This is where an intermittency route to chaos can be found. Intermittency is the phenomenon where the signal is virtually periodic except for some irregular (unpredictable) bursts. In other words, we have intermittently periodic behavior and irregular aperiodic behavior. Starting from the region of parameters where a stable period-3 limit cycle exists, as we increase the parameter R, a stable period-3 cycle emerges with an unstable period-3 cycle[2] and disappears through a bifurcation process called *tangent bifurcation*. However, a "ghost" of the period-3 cycle still remains, and the trajectory behaves for most of the time as if it was approaching a period-3 limit cycle, and intermittent behavior is observed. This is referred to as intermittency of type 1 [5]. Figs. 5(a) and (b) depict a phase portrait in the $v_{C_1} - v_{C_2}$ plane and the time waveform of v_{C_1} of a stable period-3 limit cycle. Figs. 6(a) and (b) show the phase portrait and time waveform of intermittent chaos. We see in Fig. 6(b) how the time waveform is nearly periodic of period 3 except for the appearance of irregular bursts.

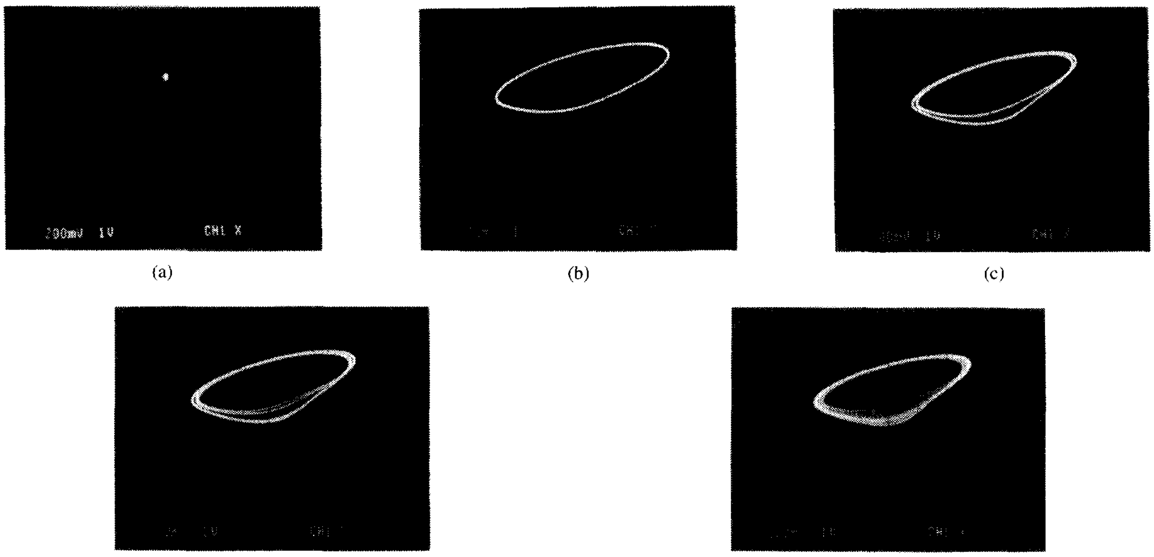
2.3. Period-Adding Bifurcations

In this section, we demonstrate the phenomenon of *period adding*, where windows of consecutive periods are separated by regions of chaos.[3] In other words, as the parameter is varied, we obtain a stable period-n orbit, $n = 1, 2, \cdots$ followed by a region of chaos, then a stable period-$(n+1)$ orbit, followed by chaos, and then a period-$(n+2)$ orbit and so on. Again, we use the same parameters as before. We have two cases here. The first case is in the spiral Chua's attractor region, where a period-3 and period-4 limit cycle are shown in Fig. 5(a) and 7(a), respectively. The second case is in the double-scroll Chua's attractor region, shown in Fig. 7(b). We show in Fig. 7(c)–(h) limit cycles in order of decreasing parameter R. Between every two neighboring region of stable limit cycles, we find a double-scroll Chua's attractor. As we decrease R further, we reach a point where a large outer limit cycle is observed (Fig. 7(i)). This is due to the fact that in any physical implementation of the Chua's diode, the $v - i$ characteristic

[1] The Andronov-Hopf bifurcation is proved for sufficiently smooth systems and, therefore, does not apply strictly to piecewise-linear systems. However, all *physical* implementations of the piecewise-linear characteristic of the nonlinear resistor is in fact smooth.

[2] The unstable limit cycle is not observable experimentally but can be shown to exist using numerical methods.

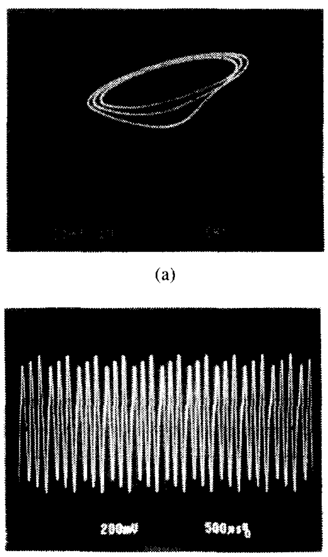
[3] The period-adding phenomenon is first observed in a physical system in [6].



(a) (b) (c)

(d) (e)

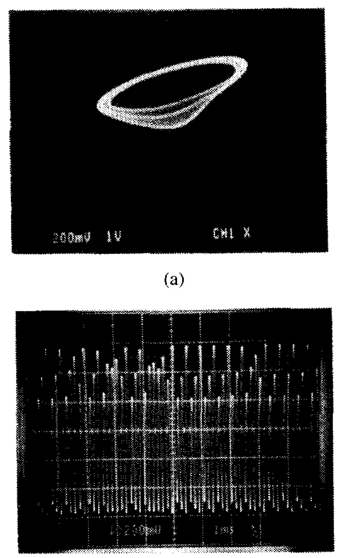
Fig. 4. Phase portraits in $v_{C_1} - v_{C_2}$ plane of a typical period-doubling sequence. $C_1 = 5.75nF$, $C_2 = 21.32nF$, $L = 12mH$, $R_0 = 30.86\Omega$, $G_a = -0.879mS$, $G_b = -0.4124mS$ and $E = 1V$: (a) Equilibrium point, $R > 1.558k\Omega$; (b) period-1 limit cycle, $R = 1.558k\Omega$; (c) period-2 limit cycle, $R = 1.516k\Omega$; (d) period-4 limit cycle, $R = 1.508k\Omega$; (e) spiral Chua's attractor, $R = 1.503k\Omega$.



(a)

(b)

Fig. 5. Intermittency near a period-3 limit cycle. Parameters are the same as in Fig. 4 except R: (a) Phase portraits in $v_{C_1} - v_{C_2}$ plane of a period-3 limit cycle, $R = 1.501k\Omega$; (b) time waveform of v_{C_1} of a period-3 limit cycle, $R = 1.501k\Omega$.



(a)

(b)

Fig. 6. (a) Intermittency of type I. Parameters are the same as in Fig. 4 except R. Phase portraits in $v_{C_1} - v_{C_2}$ plane of intermittency near a period-3 window, $R = 1.502k\Omega$; (b) time waveform of v_{C_1} of intermittency near a period-3 window, $R = 1.502k\Omega$.

must be eventually passive, even though it can be active in the region of interest. The limit cycle goes through a region in which the $v - i$ characteristic is locally passive (i.e., the slope is positive). In a computer simulation which does not take this into account, the trajectory will simply diverge to infinity.

A computer-generated bifurcation diagram which plots v_{C_1} versus the parameter $G = 1/R$ is shown in Fig. 8. We can see the periodic windows of increasing periodicity between regions of chaotic behavior.

2.4. Attractors in Chua's Oscillator Whose Only Active Element is Linear

In [7] and [8], Chua's circuit contains only one nonlinear element, which is also the only active element in the circuit.

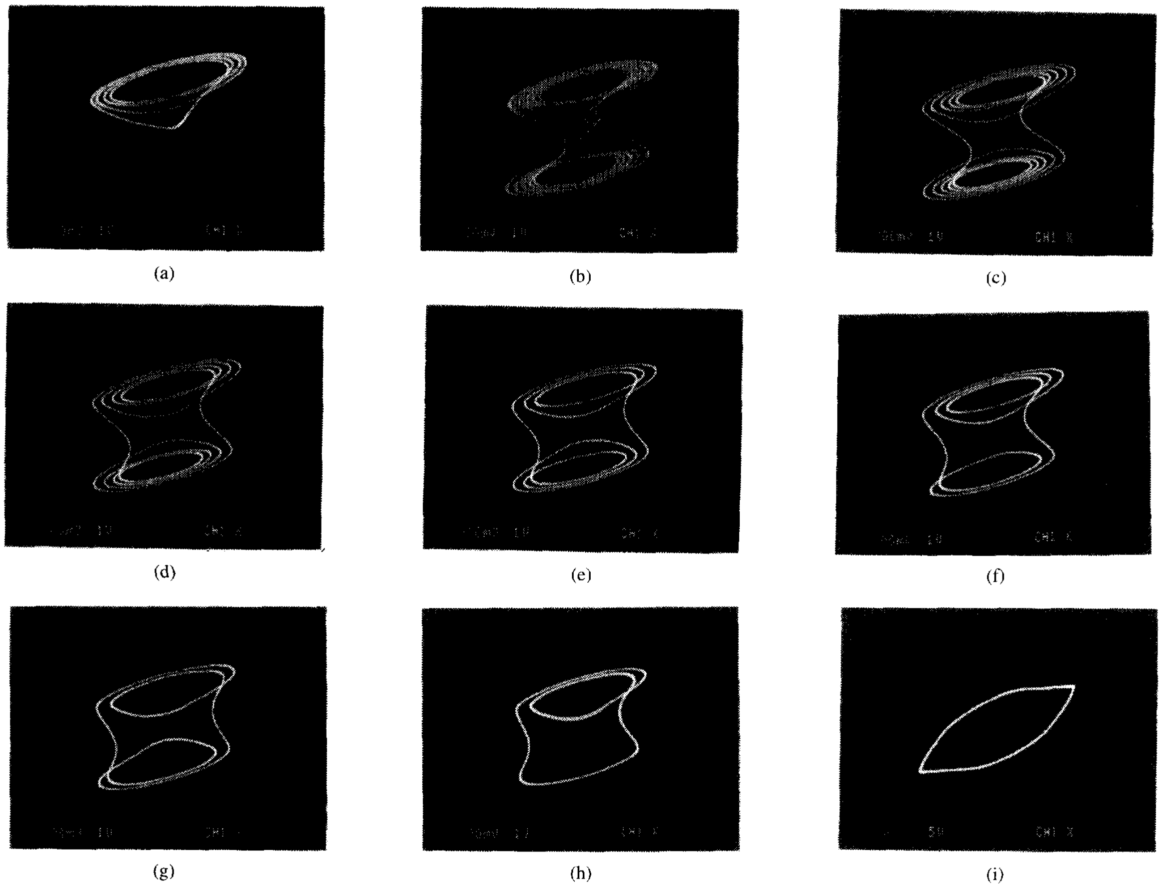


Fig. 7. Period-adding sequence. Parameters are the same as in Fig. 4 except R: (a) Period-4 limit cycle, $R = 1.493k\Omega$; (b) double scroll Chua's attractor, $R = 1.476k\Omega$; (c) $5-5$ window, $R = 1.460k\Omega$; (d) $4-4$ window, $R = 1.449k\Omega$; (e) $3-3$ window, $R = 1.430k\Omega$; (f) $3-2$ window, $R = 1.421k\Omega$; (g) $2-2$ window, $R = 1.402k\Omega$; (h) $2-1$ window, $R = 1.394k\Omega$; (i) outer periodic attractor, $R = 1.389k\Omega$.

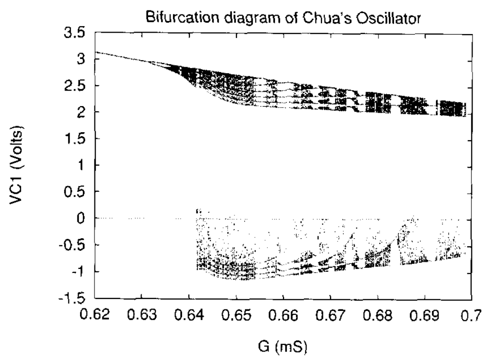


Fig. 8. Bifurcation diagram of Chua's oscillator showing period-doubling, period-3 window, and period adding phenomenon.

An active element is needed to pump energy into an otherwise purely dissipative system, in order for self-excited oscillations to occur. In this section we show a special case of Chua's oscillator where the nonlinear resistor is passive and locally passive (i.e., the v-i characteristic passes through the origin and is strictly monotone increasing), and the only active element is a linear resistor. In our experimental setup, the following parameters are fixed:

$$G_a = 0.599mS, \quad G_b = 0.77mS,$$
$$G = \frac{1}{R} = -0.7mS, \quad E = 1V. \tag{3}$$

The implementation of the nonlinear resistor is shown in Fig. 9(a) and the implementation of the negative linear resistor is shown in Fig. 9(b).

Here, we show three attractors for different parameter values from a physical circuit (Fig. 10(a)–(c)) and from computer simulations (Fig. 11(a)–(c)).

2.5. *Torus Breakdown Route to Chaos*

In the Ruelle-Takens-Newhouse route to chaos, the system undergoes several Andronov-Hopf bifurcations. After two Andronov-Hopf bifurcations, we obtain a toroidal attractor. At the third Andronov-Hopf bifurcation, chaos is likely to appear. Experimentally, this appears as a toroidal attractor bifurcating into a chaotic attractor [5]. We have obtained for certain parameter values a similar scenario in Chua's oscillator. In each of the following attractors, we will show both the phase

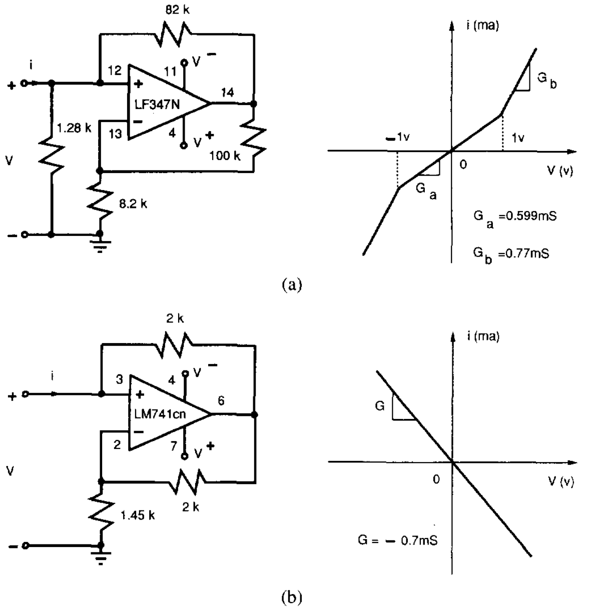


Fig. 9. (a) Implementation of nonlinear resistor N_r; (b) implementation of negative linear resistor R.

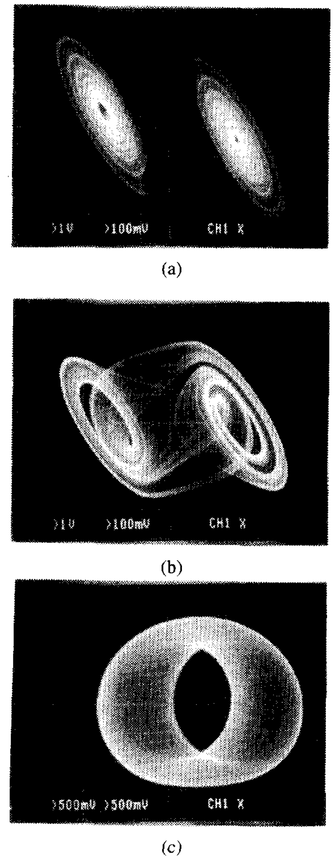


Fig. 10. Phase portrait in $v_{C_1} - v_{C_2}$ plane of experimentally observed attractors from Chua's oscillator using only one active linear element. $G_a = 0.599mS$, $G_b = 0.77mS$, $G = \frac{1}{R} = -0.7mS$, $E = 1V$: (a) $C_1 = 0.0135\mu F$, $C_2 = 1.9319\mu F$, $L = 19.003mH$, $R_0 = 26.9\Omega$; (b) $C_1 = 0.0015\mu F$, $C_2 = 0.285\mu F$, $L = 2.069mH$, $R_0 = 6.8\Omega$; (c) $C_1 = 0.0297\mu F$, $C_2 = 0.3606\mu F$, $L = 7.972mH$, $R_0 = 11.4\Omega$.

portrait and the Poincaré cross section. The fixed parameters are

$$G_a = 0.599mS, \quad G_b = 0.77mS, \quad G = \frac{1}{R} = -0.7mS, \quad E = 1V,$$

$$C_2 = 0.3406\mu F, \quad L = 7.595mH, \quad R_0 = 11.4\Omega$$

and we vary the parameter C_1. The implementations of the nonlinear resistor and negative resistor are the same as in the previous section.

First we start with an attractor resembling two tori with the trajectory jumping between them (Fig. 12(a)–(b)). As we decrease the parameter C_1, we obtain a torus attractor, whose orbit in the associated Poincaré map is a closed curve (Fig. 12(c)–(d)). As we decrease the parameter C_1 further, we go through a period-adding sequence of periodic windows of *consecutively* decreasing periods. Between the periodic windows we find torus attractors. Here we show a period-16 limit cycle (Fig. 12(e)–(f)), a torus attractor (Fig. 12(g)–(h)), a period-15 limit cycle (Fig. 12(i)–(j)), and a period-14 limit cycle (Fig. 12(k)–(l)). If we decrease the parameter C_1 further, we observe a period-doubling sequence ending in a chaotic attractor. We show a period-4 (Fig. 12(m)–(n)), a period-8 limit cycle (Fig. 12(o)–(p)) and a chaotic attractor (Fig. 12(q)–(r)). In the corresponding Poincaré maps the closed curve earlier associated with a torus (e.g., Fig. 12(h)) now begins to "deform" and starts to wrinkle and develop "folds" (Fig. 12(r)). As C_1 is further decreased, we obtain a chaotic attractor similar to the double-scroll Chua's attractor (Fig. 12(s)∼(t)).

A computer generated bifurcation diagram which plot v_{C_1} versus C_1 is shown in Fig. 13. See [9] for more details on the torus breakdown route.

III. Universality of Chua's Oscillator

In this section, we give a result that shows that the vector field of Chua's oscillator is *topologically conjugate*[4] to the vector field of a large class of 3-D piecewise-linear vector fields. In the companion paper [1], we illustrate how any vector field in this class can be mapped into parameters for the Chua's oscillator by means of several examples.

By a change of variables, the state equations of Chua's oscillator (1) can be transformed into the following dimensionless

[4]The vector fields f and g are said to be topologically conjugate if there exists a continuous map h with a continuous inverse such that h maps orbits of f into orbits of g preserving time orientation and parametrization of time [10]. If ϕ_t and ψ_t are flows of f and g respectively, then $\phi_t \circ h = h \circ \psi_t$ for all t. This means that the dynamics of f and g are qualitatively the same.

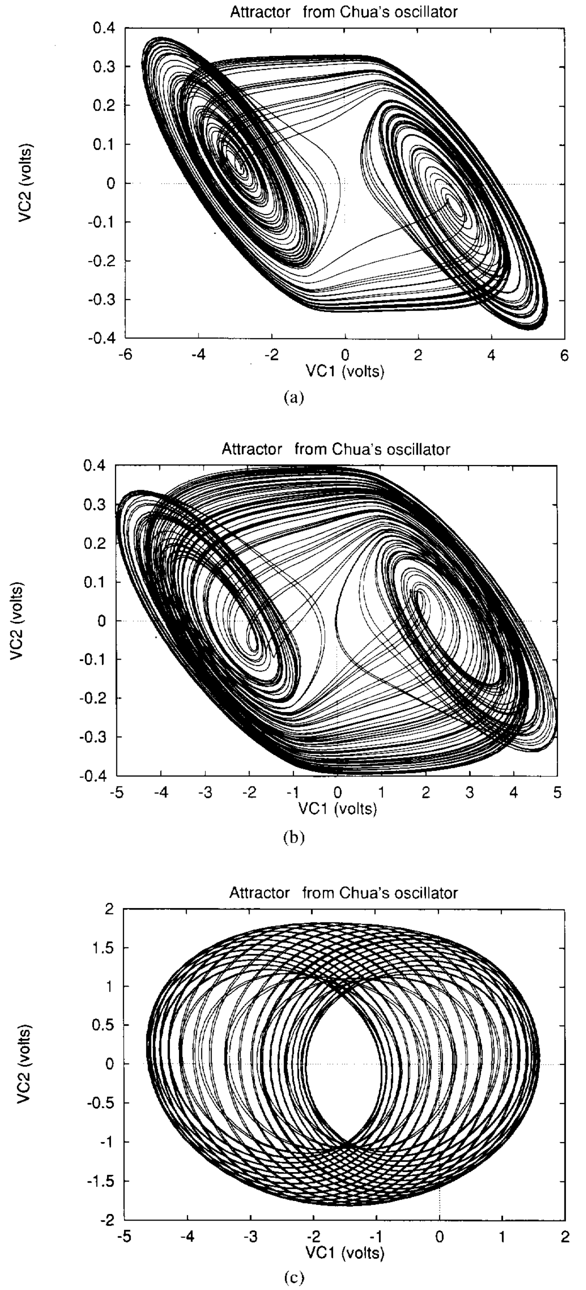


Fig. 11. Computer generated phase portrait in $v_{C_1} - v_{C_2}$ plane of attractors in Fig. 10. $G_a = 0.599mS, G_b = 0.77mS, G = \frac{1}{R} = -0.7mS, E = 1V$: (a) $C_1 = 0.0135\mu F$, $C_2 = 1.9319\mu F$, $L = 19mH$, $R_0 = 26.7\Omega$; (b) $C_1 = 0.019\mu F$, $C_2 = 0.285\mu F$, $L = 2.056mH$, $R_0 = 12.5\Omega$; (c)$C_1 = 0.0297\mu F$, $C_2 = 0.3606\mu F$, $L = 7.682mH$, $R_0 = 13.4\Omega$.

form:

$$\left.\begin{aligned} \frac{dx}{d\tau} &= k\alpha(y - x - f(x)) \\ \frac{dy}{d\tau} &= k(x - y + z) \\ \frac{dz}{d\tau} &= k(-\beta y - \gamma z) \\ f(x) &= bx + \tfrac{1}{2}(a-b)\{|x+1| - |x-1|\} \end{aligned}\right\} \qquad (4)$$

where

$$\left.\begin{aligned} & x \triangleq \tfrac{v_1}{E}, \quad y \triangleq \tfrac{v_2}{E}, \quad z \triangleq i_3\left(\tfrac{R}{E}\right) \\ & \alpha \triangleq \tfrac{C_2}{C_1}, \quad \beta \triangleq \tfrac{R^2 C_2}{L}, \quad \gamma \triangleq \tfrac{R R_0 C_2}{L} \\ & a \triangleq RG_a, \quad b \triangleq RG_b, \quad \tau \triangleq \tfrac{t}{|RC_2|}, \\ & \quad \text{and} \\ & k = 1, \quad \text{if } RC_2 > 0 \\ & k = -1, \quad \text{if } RC_2 < 0 \end{aligned}\right\} \qquad (5)$$

We will work primarily with the dimensionless form in this section. Note that there are more than one set of circuit parameters (C_1, C_2, etc.) that maps onto the same dimensionless equations (4). Furthermore, by selecting the constant RC_2 we determine how "fast" the real circuit is in comparison with the dimensionless system.

3.1. Topological Conjugacy

Here we give the theorem of topological conjugacy proved in [11]. First we define the Class $\mathcal{C}$ of vector fields in $\mathbb{R}^3$.

Definition 1: A dynamical system defined by a state equation

$$\dot{x} = \boldsymbol{f}(x), \quad x \in \mathbb{R}^3 \qquad (6)$$

is said to belong to Class $\mathcal{C}$ iff

1. $\boldsymbol{f}(\cdot)$ is continuous
2. $\boldsymbol{f}(\cdot)$ is odd-symmetric, i.e.,

$$\boldsymbol{f}(x) = -\boldsymbol{f}(-x)$$

3. $\mathbb{R}^3$ is partitioned by 2 parallel boundary planes U_1 and U_{-1} into an inner region D_0 containing the origin, and two outer regions D_1 and D_{-1}, and $\boldsymbol{f}(\cdot)$ is affine in each region.[5]

Without loss of generality, the boundary planes U_1 and U_{-1} can be chosen to be

$$U_1 : x_1 = 1 \qquad (7)$$

$$U_{-1} : x_1 = -1 \qquad (8)$$

Then any vector field in the family $\mathcal{C}$ can be represented as

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}, \qquad x_1 \geq 1 \qquad (9a)$$

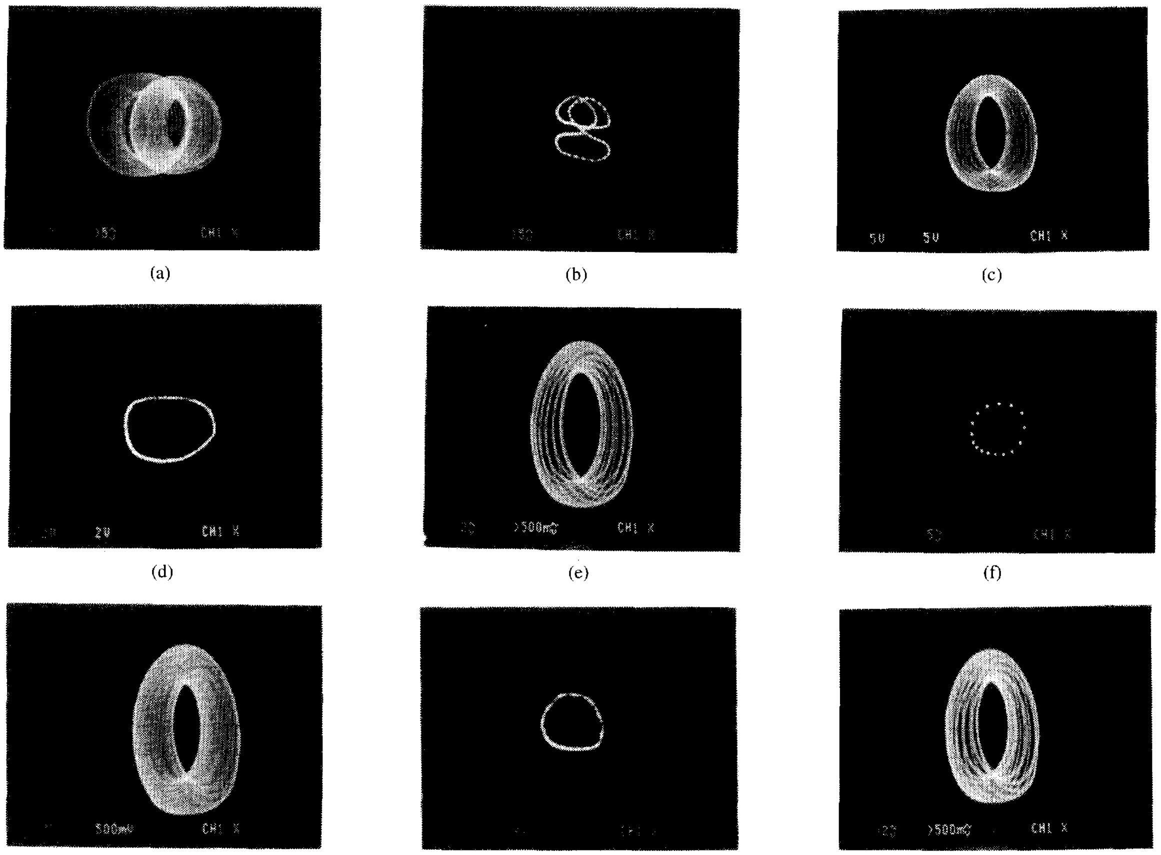
$$= \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}, \qquad x_1 \leq -1 \qquad (9b)$$

$$= \boldsymbol{A}_0\boldsymbol{x}, \qquad -1 \leq x_1 \leq 1 \qquad (10)$$

where

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \qquad (11)$$

[5] By condition 2, the vector field in D_0 must be linear, i.e., it is zero at the origin.



(a) (b) (c) (d) (e) (f) (g) (h) (i)

Fig. 12. Phase portraits and Poincaré maps of torus breakdown sequence. Vertical axis is v_{C_2} and horizonal axis is v_{C_1}. $G_a = 0.599mS$, $G_b = 0.77mS$, $G = \frac{1}{R} = -0.7mS$, $E = 1V$, $C_2 = 0.3406\mu F$, $L = 7.595mH$, $R_0 = 11.4\Omega$: (a) Phase portrait: $C_1 = 0.0221\mu F$; (b) Poincaré map: $C_1 = 0.0221\mu F$; (c) phase portrait: $C_1 = 0.0217\mu F$; (d) Poincaré map: $C_1 = 0.0217\mu F$; (e) phase portrait: $C_1 = 0.0213\mu F$; (f) Poincaré map: $C_1 = 0.0213\mu F$; (g) phase portrait: $C_1 = 0.0205\mu F$; (h) Poincaré map: $C_1 = 0.0205\mu F$; (i) phase portrait: $C_1 = 0.0192\mu F$.

Because of continuity,

$$A_0 = A + \begin{bmatrix} b_1 & 0 & 0 \\ b_2 & 0 & 0 \\ b_3 & 0 & 0 \end{bmatrix} \tag{12}$$

Let (μ_1, μ_2, μ_3) denote the eigenvalues associated with the linear vector field in the region D_0 and let (ν_1, ν_2, ν_3) denote the eigenvalues associated with the affine vector field in the regions D_1 and D_{-1}.

We define

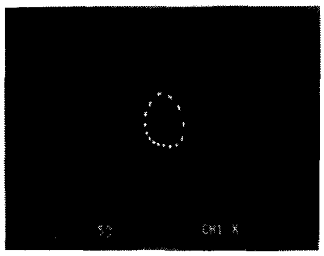
$$\left.\begin{array}{ll} p_1 = \mu_1 + \mu_2 + \mu_3 & q_1 = \nu_1 + \nu_2 + \nu_3 \\ p_2 = \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1 & q_2 = \nu_1\nu_2 + \nu_2\nu_3 + \nu_3\nu_1 \\ p_3 = \mu_1\mu_2\mu_3 & q_3 = \nu_1\nu_2\nu_3 \end{array}\right\} \tag{13}$$

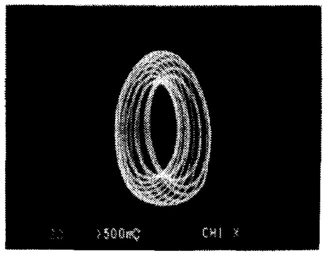
Since $\{p_1, p_2, p_3; q_1, q_2, q_3\}$ are *uniquely* determined by the eigenvalues $\{\mu_1, \mu_2, \mu_3; \nu_1, \nu_2, \nu_3\}$ and vice versa, we call these the "equivalent eigenvalue parameters." These are easier to work with than the eigenvalues since they are all real numbers and are just the coefficients of the characteristic polynomial:

$$(s-\mu_1)(s-\mu_2)(s-\mu_3) = s^3 - p_1 s^2 + p_2 s - p_3 \tag{14}$$
$$(s-\nu_1)(s-\nu_2)(s-\nu_3) = s^3 - q_1 s^2 + q_2 s - q_3 \tag{15}$$

Theorem 1: Let $\{\mu_1, \mu_2, \mu_3, \nu_1, \nu_2, \nu_3\}$ be the eigenvalues associated with a vector field $\boldsymbol{F}(\boldsymbol{x}) \in \mathcal{C}\backslash\mathcal{E}_0$, where $\mathcal{E}_0$ is the set of measure zero in the space of equivalent eigenvalue parameters where one of (16)–(20) is satisfied. Then, Chua's oscillator with parameters defined by (22)–(23) is linearly conjugate (i.e., topologically conjugate through a linear map) to this vector field.

$$p_1 - q_1 = 0 \tag{16}$$
$$p_2 - \frac{q_3 - p_3}{q_1 - p_1} + \left(\frac{p_2 - q_2}{q_1 - p_1}\right)\left(\frac{p_2 - q_2}{q_1 - p_1} + p_1\right) = 0 \tag{17}$$
$$\frac{p_2 - q_2}{q_1 - p_1} - \frac{k_1}{k_2} = 0 \tag{18}$$
$$-k_1 k_3 + k_2\left(\frac{p_3 - q_3}{p_1 - q_1}\right) = 0 \tag{19}$$


(j)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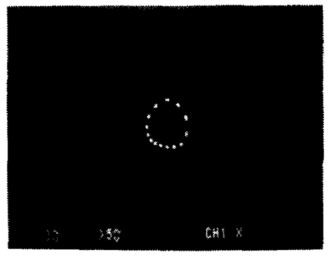
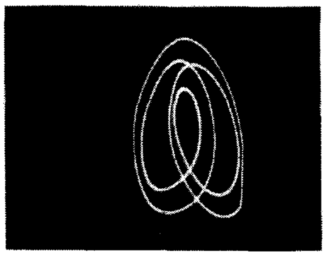
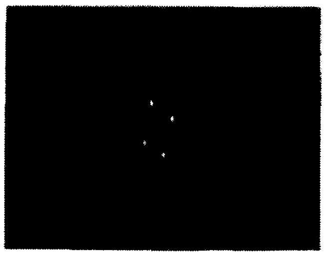
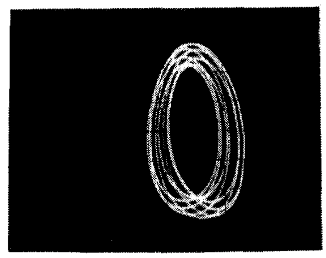
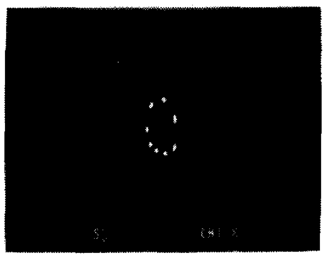
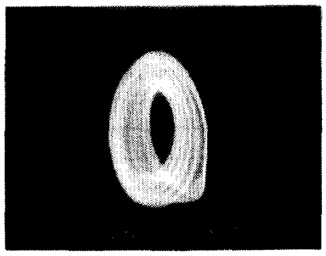

(l)

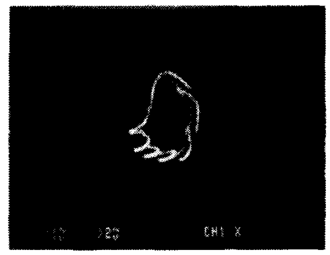

(m)

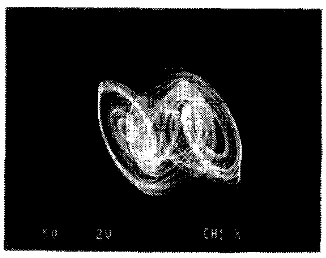

(n)


(o)


(p)


(q)


(r)


(s)


(t)

Fig. 12. Phase portraits and Poincaré maps of torus breakdown sequence. Vertical axis is v_{C_2} and horizonal axis is v_{C_1}. $G_a = 0.599mS$, $G_b = 0.77mS$, $G = \frac{1}{R} = -0.7mS$, $E = 1V$, $C_2 = 0.3406\mu F$, $L = 7.595mH$, $R_0 = 11.4\Omega$: (j) Poincaré map: $C_1 = 0.0192\mu F$; (k) phase portrait: $C_1 = 0.0176\mu F$; (l) Poincaré map: $C_1 = 0.0176\mu F$; (m) phase portrait: $C_1 = 0.0104\mu F$; (n) Poincaré map: $C_1 = 0.0104\mu F$; (o) phase portrait: $C_1 = 0.0098\mu F$; (p) Poincaré map: $C_1 = 0.0098\mu F$; (q) phase portrait: $C_1 = 0.0093\mu F$; (r) Poincaré map: $C_1 = 0.0093\mu F$; (s) phase portrait: $C_1 = 0.0087\mu F$; (t) Poincaré map: $C_1 = 0.0087\mu F$.

$$\det \boldsymbol{K} = \det \begin{bmatrix} 1 & 0 & 0 \\ a_{11} & a_{12} & a_{13} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} = a_{12}K_{33} - a_{13}K_{32} = 0 \tag{20}$$

where

$$K_{3i} \triangleq \sum_{j=1}^{3} a_{1j}a_{ji}, \qquad i = 1, 2, 3 \tag{21}$$

The proof of this theorem relies on the fact that two vector fields in $\mathcal{C}$ are linearly conjugate if they have the same eigenvalues in each region. We shall denote $\tilde{\mathcal{C}} = \mathcal{C} \backslash \mathcal{E}_0$. The algorithm for finding the parameters that make a Chua's oscillator topologically conjugate to a particular vector field in $\tilde{\mathcal{C}}$ is as follows:

Algorithm 1:

1. Calculate the eigenvalues (μ_1', μ_2', μ_3'), and (ν_1', ν_2', ν_3') associated with the linear and affine vector fields, respectively, of the circuit or system candidate whose *attractor* is being *reproduced* by Chua's oscillator, up to a linear conjugacy.

2. Find a set of circuit parameters $\{C_1, C_2, L, R, R_0, G_a, G_b, E\}$ (or dimensionless parameters $\{\alpha, \beta, \gamma, a, b, k\}$) so that the resulting eigenvalues μ_j, ν_j for Chua's oscillator satisfy $\mu_j = \mu'_j$ and $\nu_j = \nu'_j$, $j = 1, 2, 3$.

The formula for doing step 2 is given by:[6]

$$\left.\begin{aligned} C_1 &= 1 \\ C_2 &= -\frac{k_2}{k_3^2} \\ L &= -\frac{k_3^2}{k_4} \\ R &= -\frac{k_3}{k_2} \\ R_0 &= -\frac{k_1 k_3^2}{k_2 k_4} \\ G_a &= -p_1 - \left(\frac{p_2-q_2}{p_1-q_1}\right) + \frac{k_2}{k_3} \\ G_b &= -q_1 - \left(\frac{p_2-q_2}{p_1-q_1}\right) + \frac{k_2}{k_3} \end{aligned}\right\} \qquad (22)$$

where $\{p_1, p_2, p_3, q_1, q_2, q_3\}$ are the "equivalent eigenvalue parameters" defined in (13), and

$$\left.\begin{aligned} k_1 &\triangleq -p_3 + \left(\frac{q_3-p_3}{q_1-p_1}\right)\left(p_1 + \frac{p_2-q_2}{q_1-p_1}\right) \\ k_2 &\triangleq p_2 - \left(\frac{q_3-p_3}{q_1-p_1}\right) + \left(\frac{p_2-q_2}{q_1-p_1}\right)\left(\frac{p_2-q_2}{q_1-p_1} + p_1\right) \\ k_3 &\triangleq \left(\frac{p_2-q_2}{q_1-p_1}\right) - \frac{k_1}{k_2} \\ k_4 &\triangleq -k_1 k_3 + k_2\left(\frac{p_3-q_3}{p_1-q_1}\right) \end{aligned}\right\}. \qquad (23)$$

The breakpoint of the piecewise-linear Chua's diode, E, can be chosen arbitrarily as it will not affect the eigenvalues nor the dynamics in each region.

In terms of the dimensionless parameters, the formulas are:[7]

$$\left.\begin{aligned} \alpha &= \frac{1}{l_1 l_3^2} \\ \beta &= 1 + \frac{1}{l_1^2 l_3^2}\frac{p_3-q_3}{p_1-q_1} - \frac{-p_2+q_2}{p_1-q_1}\frac{1}{l_1 l_3} \\ \gamma &= -1 + \frac{-p_2+q_2}{p_1-q_1}\frac{1}{l_1 l_3} \\ a &= -1 - \left(p_1 + \frac{-p_2+q_2}{p_1-q_1}\right) l_3 \\ b &= -1 - \left(q_1 + \frac{-p_2+q_2}{p_1-q_1}\right) l_3 \\ l &= \operatorname{sgn}(l_1 l_3) \end{aligned}\right\} \qquad (24)$$

where

$$\left.\begin{aligned} l_1 &= -p_2 - \left(\frac{p_2-q_2}{p_1-q_1} - p_1\right)\frac{p_2-q_2}{p_1-q_1} + \frac{p_3-q_3}{p_1-q_1} \\ l_2 &= -p_3 + \frac{p_3-q_3}{p_1-q_1}\left(p_1 + \frac{-p_2+q_2}{p_1-q_1}\right) \\ l_3 &= \frac{-p_2+q_2}{p_1-q_1}\frac{1}{l_1} + \frac{l_2}{l_1^2} \end{aligned}\right\}. \qquad (25)$$

IV. Brief History and Selected Bibliography on Chua's Circuit

The chaotic nature of Chua's circuit was first observed by Matsumoto in 1983 using computer simulations [12], following the instructions of Chua, who had invented this circuit and had explained its operating principles to Matsumoto moments before he was rushed to a hospital for a major surgery, and who did not participate in the early phases of this research. In acknowledging his subsidiary role as a computer programmer, Matsumoto had named this circuit *Chua's circuit* [7], [12]. The first experimental Chua's circuit which confirms the presence of chaos was due to Zhong and Ayrom in 1984 [8], [13]. A second experimental circuit was reported by Matsumoto shortly after [14] and was designed by Tokunaga, who is also responsible for obtaining all of the experimental result presented in that paper. The global bifurcation landscape of Chua's circuit [15] was obtained by Komuro and a team of students of Matsumoto. The colorful bifurcation landscape in this paper [15] (shown copyrighted by Matsumoto in Figs. 17, 20, and 22) was drawn by a professional artist. The first rigorous proof of the chaotic nature of Chua's circuit was given in [16], where the authors proved that there exists some parameters (α, β) such that Chua's circuit satisfies Shil'nikov's theorem and, therefore, has infinitely many Horseshoe maps [22]. Although the authors in [16] are listed in alphabetical order as a compromise to Matsumoto's tradition of ordering his name first in earlier publications on Chua's circuit, the rigorous proof of the main theorem is due to Komuro. However, since the limiting Cantor set from a Horseshoe map is *not* an attractor, this result does not imply that the double scroll Chua's attractor is directly related to the chaotic phenomena associated with the Horseshoe map. This unsatisfactory situation has now been resolved by a recent proof that a 2-D geometrical model of Chua's circuit gives rise to a *double Horseshoe map* which generates strange attractors [17]. Another milestone was achieved in 1990 when a canonical circuit was discovered which is qualitative equivalent to a 21-parameter family $\mathcal{C}$ of continuous odd-symmetric piecewise-linear vector fields [18]. Inspired by a question posed by professor J. Neirynck in 1991[8] on whether this canonical circuit is unique, a systematic search has since been completed by several researchers, including A. Huang and Lj. Kocarev, where many more distinct canonical circuits has been found. The universal circuit presented in this paper is therefore only one among many qualitatively equivalent circuits in the class $\mathcal{C}^*$.[9] Since the circuit in Fig. 2 is a direct generalization of Chua's circuit, it is logical to

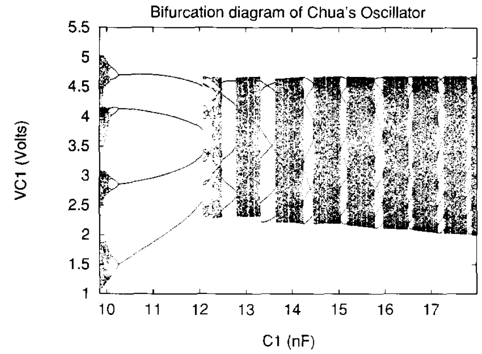


Fig. 13. Bifurcation diagram of torus breakdown sequence.

[6] When building a physical circuit, these parameters need to be scaled appropriately to reasonable values.

[7] In the dimensionless form, the eigenvalues are normalized. This corresponds to a scaling in time which perserves the dynamics up to topological equivalency, which is weaker than topological conjugacy in that time scales are not preserved.

[8] This question was raised during a seminar on the canonical circuit in [18] given by Chua.

[9] This class is defined in the companion paper [1].

choose this circuit in future studies of the family $\mathcal{C}^*$.[10] The name Chua's Oscillator was given by Madan [3], [4] in order to distinguish this "globally unfolded" circuit [11] from the original Chua's circuit.

Next we give a selected bibliography on Chua's circuit and Chua's oscillator for future researchers of these circuits.

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V. Concluding Remarks

Virtually every bifurcation and chaotic phenomena that have been reported in the literature from systems modelled by autonomous third-order ordinary differential equations have been observed from Chua's circuit, which is a special case of Chua's oscillator with $R_0 = 0$. Moreover, it remains the only known chaotic system where in-depth *theoretical*, *experimental*, and *numerical* investigations conducted by many international research groups have all provided consistent results. In contrast, research from other disciplines are often restricted to only one or perhaps two of these three complementary methods of investigation because the associated physical systems (e.g., plasma, turbulence, hydrodynamics, chemical reactions, etc.) often do not have realistic and/or mathematically tractable models.

Consequently, for those readers interested in learning all three aspects (theoretical, experimental, and numerical) of bifurcation and chaos in physical systems, it suffices to study Chua's circuit. An excellent sophomore level introduction to this circuit can be found in [19], [20]. For more advanced and current research results, the reader is referred to [21] for a compendium of 56 recent research papers on this subject, the *Proceedings of the International Symposium on Nonlinear Theory and Applications (NOLTA'93)*, [22], and [23].

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